General Relativity: From Christoffel Symbols to the Riemann Tensor

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The goal of this talk is to derive a means of calculating the intrinsic curvature of a manifold: the Riemann Curvature Tensor. We begin with an introduction to covariant differentiation, showing that since the basis vectors of a coordinate system change, we must utilize a correction term to obtain a derivative-like formulation for covariant vectors, known as the covariant derivative: $\nabla_r V_m = \partial_r V_m - \Gamma^t_{rm} V_t$. Much like derivatives are the generators of infinitesimal translations, covariant derivatives generate infinitesimal parallel transport of vectors. We subsequently derive the equation for the Christoffel symbols, assuming a torsion-free physical system. By exploiting the symmetry of the covariant indices, we find that $\Gamma^t_{mn} = \frac{1}{2} g^{rt}[\partial_n g_{rm} + \partial_m g_{rn} - \partial_r g_{mn}]$. We also discuss the covariant derivative of a mixed tensor, $\nabla_\mu T^\lambda_\nu = \partial_\mu T^\lambda_\nu + \Gamma^\lambda_{\alpha\mu} T^\alpha_\nu - \Gamma^\sigma_{\mu\nu} T^\lambda_\sigma$. By combining $\nabla_\mu T^\lambda_\nu$ and $\nabla_r T_m$, we obtain the Riemann Curvature Tensor, $R^t_{srn} = \partial_r \Gamma^t_{sn} - \partial_s \Gamma^t_{rn} + \Gamma^p_{sn} \Gamma^t_{pr} - \Gamma^p_{rn} \Gamma^t_{ps}$. We summarize this result in the context of General Relativity, and demonstrate the four steps to deriving the metric for a given spacetime: calculate the Christoffel symbols, substitute it into the Riemann curvature tensor, contract it into the Ricci tensor, and finallly substitute it into Einstein's field equation.

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